Lecture 9: Tangents and Normal to Curves

Tangent Vector

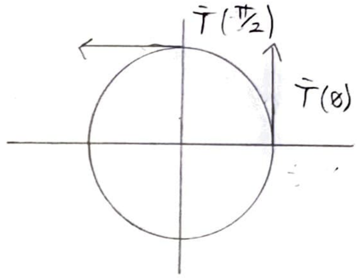
The tangent vector, $\bar{T}(t)$, of a smooth curve is given by:

$$\bar{T}(t) = \frac{\bar{r}'(t)}{|\bar{r}'(t)|} = \frac{\frac{d\bar{r}}{dt}}{\left|\frac{dr}{dt}\right|}$$

Ex. 1 Find the tangent vector of $\bar{r} = \langle a\cos(t), b\sin(t), 0 \rangle$ at the point $(1, 0, 0)$ and $(0, 1, 0)$

$\bar{r}'(t) = \langle -a\sin(t), a\cos(t), 0 \rangle$

$|\bar{r}'(t)| = a$



$\bar{T}(t) = \langle -\sin(t), \cos(t), 0 \rangle$

$(1,0,0) \rightarrow t = 0$ $\quad$ $(0,1,0) \rightarrow t = \pi/2$

$\bar{T}(t=0) = \langle 0, 1, 0 \rangle$ $\quad$ $\bar{T}(t=\pi/2) = \langle -1, 0, 0 \rangle$

Curvature

The curvature of a smooth curve is given by

$$K(t) = \frac{|\bar{T}'(t)|}{|\bar{r}'(t)|}$$

Ex. 2 Find the curvature of $\bar{r}(t) = \langle a\cos(t), a\sin(t), 0 \rangle$

$\bar{T}'(t) = \langle -\cos(t), -\sin(t), 0 \rangle$

$|\bar{T}'(t)| = 1$

$K(t) = \frac{1}{a}$ a constant!

Normal Vector

$\bar{T} \cdot \bar{T} = ||T|| = 1$

$\frac{d}{dt}(\bar{T} \cdot \bar{T}) = 0$

$2(\bar{T}' \cdot T) = 0$

$\bar{T}' \perp \bar{T}$

We define the Normal vector

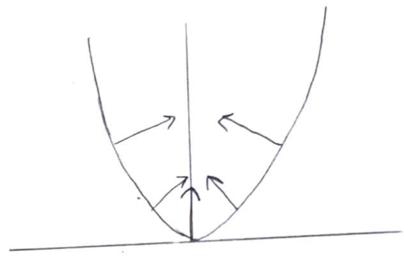
$$\bar{N}(t) = \frac{\bar{T}'(t)}{|\bar{T}'(t)|}$$

Ex. 3 Find the unit Normal to $\bar{r}(t) = \langle a\cos(t), a\sin(t), \emptyset \rangle$

$$\bar{N}(t) = \frac{\langle -\cos(t), -\sin(t), \emptyset \rangle}{1} = -\frac{1}{a}\bar{r}(t)$$

Ex. 4 Find the tangent and normal vector of $x = y^2$

$\bar{r}(t) = \langle t, t^2, \emptyset \rangle$



Normal vector

$\bar{r}'(t) = \langle 1, 2t, \emptyset \rangle$

$|\bar{r}'(t)| = \sqrt{1^2 + 4t^2}$

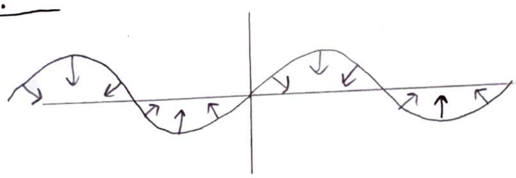
$$\bar{T}(t) = \frac{1}{\sqrt{1+4t^2}} \langle 1, 2t, \emptyset \rangle$$

$$\bar{T}'(t) = \left\langle \frac{-4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}}, \emptyset \right\rangle$$

$$|\bar{T}'(t)| = \frac{2}{1+4t^2}$$

$$\bar{N}(t) = \frac{1}{\sqrt{1+4t^2}} \langle -2t, 1 \rangle$$

Ex. 5



Tangential and Normal Components of acceleration

$$\bar{v} = \frac{d\bar{r}}{dt} = \left|\frac{d\bar{r}}{dt}\right| \bar{T} = |\bar{v}|\,\bar{T}$$

$$\frac{d\bar{v}}{dt} = \frac{d|\bar{v}|}{dt}\bar{T} + |\bar{v}|\frac{d\bar{T}}{dt}$$

$$\bar{a} = \frac{d|\bar{v}|}{dt} + |\bar{v}| \frac{d\bar{T}}{dt}$$

$$\bar{a} = \frac{d|\bar{v}|}{dt}\bar{T} + |v| \left|\frac{d\bar{T}}{dt}\right| \bar{N}$$

$$\bar{a} = a_T \bar{T} + a_N \bar{N} \quad , \quad a_T = \frac{d|\bar{v}|}{dt} \;\&\; a_N = |\bar{v}| \left|\frac{d\bar{T}}{dt}\right|$$

Tangential component   Normal component

$$a_N = \sqrt{|\bar{a}|^2 - a_T^2}$$

Also,

$$a_T = \frac{\bar{v} \cdot \bar{a}}{|\bar{v}|} \quad \& \quad a_N = \frac{|\bar{v} \times \bar{a}|}{|\bar{v}|}$$

Ex. 8 Find a_T & a_N at $t = 1$ for $\bar{r}(t) = \langle 2t, t^2, \frac{1}{3}t^3 \rangle$

$$\bar{r}' = \langle 2, 2t, t^2 \rangle = \bar{v} \qquad |\bar{v}| = \sqrt{2^2 + 4t^2 + t^4}$$

$$\bar{r}'' = \langle 0, 2, 2t \rangle = \bar{a} \qquad |\bar{a}| = \sqrt{4 + 4t^2} = 2\sqrt{1 + t^2}$$

$$\bar{v} \cdot \bar{a} = \langle 2, 2t, t^2 \rangle \cdot \langle 0, 2, 2t \rangle = 0 + 4t + 2t^3$$

$$a_T = \frac{4t + 2t^3}{\sqrt{4 + 4t^2 + t^4}} \qquad a_T\Big|_{t=1} = \frac{6}{3} = \boxed{2 = a_T\Big|_{t=0}}$$

$$\bar{v} \times \bar{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2t & t^2 \\ 0 & 2 & 2t \end{vmatrix} = \hat{i}(4t^2 - 2t^2) + -\hat{j}(4t - 0) + \hat{k}(4 - 0) = \langle 2t^2, -4t, 4 \rangle$$

$$|\bar{v} \times \bar{a}| = \sqrt{4t^4 + 16t^2 + 16} = 2\sqrt{(4 + 4t^2 + t^4)}$$

$$a_N = 2\sqrt{\frac{4 + 4t^2 + t^4}{4 + 4t^2 + t^4}} \qquad a_N\Big|_{t=0} = 2\sqrt{\frac{4+4+1}{4+4+1}} = \boxed{2 = a_N\Big|_{t=0}}$$