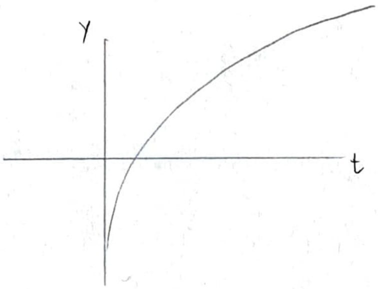
Lecture 8 Curves and Lengths

A "curve" is the range of a continuous VVF on an interval in $\mathbb{R}$.

Ex. 1 $r(t) = t\hat{i} + \ln(t)\hat{j}$

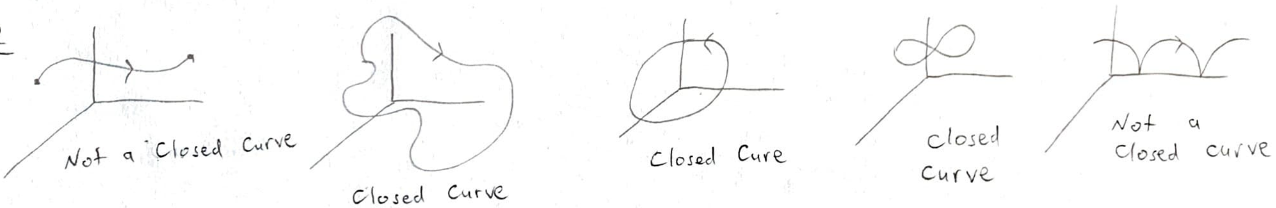
This is continuous on $(\emptyset, \infty)$. The curve is the range, the "graph" we draw.



Closed Curves

A curve is closed if it has a parameterization on the interval $[a,b]$ such that $r(a) = r(b)$ and there are at most finite overlaps.

Ex. 2

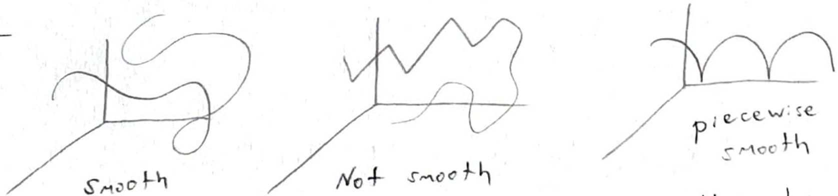


Smooth Curves

A VVF, $\bar{r}$, defined on an interval, I, is smooth if $\bar{r}$ has a continuous derivative and $\bar{r}'(t) \neq \emptyset$ $\forall t \in I$. A curve is smooth if it has a smooth parameterization.

A VVF, $\bar{r}$, is piecewise smooth if it can be divided into finitely many smooth parts. A curve is piecewise smooth if it has a piecewise smooth parameterization.

Ex. 3



Ex. 4 Find a smooth parameterization of the line segment from $(4,3,5)$ and $(2,8,5)$.

$$\bar{r}(t) = \langle x_0 + (x_1 - x_0)t,\ y_0 + (y_1 - y_0)t,\ z_0 + (z_1 - z_0)t \rangle$$

$$\bar{r}(t) = \langle 4 - 2t,\ 3 + 5t,\ 5 \rangle$$

Length of a Curve

The length of a Curve is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$L = \int_a^b |r'(t)|\, dt$$

Ex. 5 Find the length of the curve given by $\bar{r}(t) = \langle \cos(t), \sin(t), t \rangle$ from $(1, \emptyset, \emptyset)$ to $(1, \emptyset, 2\pi)$.

$\bar{r}(t) = \langle 1, \emptyset, \emptyset \rangle$ when $t = \emptyset$

$\bar{r}(t) = \langle 1, 0, 2\pi \rangle$ when $t = 2\pi$

$$\bar{r}'(t) = \langle -\sin(t), \cos(t), 1 \rangle$$

$$|\bar{r}'(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2} = \sqrt{2}$$

$$L = \int_0^{2\pi} \bar{r}'(t)\, dt = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi$$

Arc Length Function

We also define an arc length function,

$$s(t) = \int_0^t |\bar{r}'(u)|\, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

The derivative of $s(t)$ is also often useful

$$\frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$\frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right| = |v(t)|$$

Ex. 6 Reparameterize $\bar{r}(t) = \langle \cos(t), \sin(t), t \rangle$ in terms of the arc length starting from $(1, \emptyset, \emptyset)$.

$(1, \emptyset, \emptyset) \rightarrow t = \emptyset$

$$\frac{ds}{dt} = |\bar{r}'(t)| = \sqrt{2}$$

$$s(t) = \int_0^t |\bar{r}'|\,dt = \sqrt{2}\,t$$

Thus $t = \frac{s}{\sqrt{2}}$ or $t(s) = \frac{s}{\sqrt{2}}$

$$r(t(s)) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle$$