Lecture 4: Lines

We define lines in 3 main ways.

Parametric Equations

A line is defined by a single point, P_0, and a vector, $\bar{L} = \langle a, b, c \rangle$. We can then write 3 eqs.

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct.$$

Here, t is a "parameter" and it "parameterizes" the line. Any real number can be plugged in for t and a corresponding point emerges.

Ex. 1 Find the parametric equations of a line passing through the point $(3, -1, 5)$ and parallel to $\langle -1, 4, 2 \rangle$

$$P_0 = (3, -1, 5) \qquad \bar{L} = \langle -1, 4, 2 \rangle$$

$$x = 3 - t \qquad y = -1 + 4t \qquad z = 5 + 2t$$

Vector Equation

We can also write the equation of a line by combining the components from the parametric equations into a vector:

$$\bar{r} = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

$$\bar{r} = \langle x_0, y_0, z_0 \rangle + \langle a, b, c \rangle t$$

$$\bar{r} = \bar{r}_0 + t\bar{L}$$

Ex. 2 Write the vector equation of the line from the previou example

$$\bar{r}_0 = \langle x_0, y_0, z_0 \rangle = \langle 3, -1, 5 \rangle \qquad \bar{L} = \langle a, b, c \rangle = \langle -1, 4, 2 \rangle$$

$$\bar{r} = \langle 3, -1, 5 \rangle + t\langle -1, 4, 2 \rangle$$

Symmetric Equations

It is sometimes useful to solve the parametric equations for t and set them all equal

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = t$$

Ex. 3 Find the para. & symm eqs. of a line passing through $P=(2,4,-3)$ and $Q=(3,-1,1)$

$$\overline{PQ} = \langle 3-2, -1-4, 1-^{-}3\rangle = \langle 1,-5,4\rangle$$

$$\bar{r} = \bar{P} + t\overline{PQ}$$

$$\bar{r} = \langle 2,4,-3\rangle + t\langle 1,-5,4\rangle$$

$$x = 2+t \qquad y = 4-5t \qquad z = -3+4t$$

$$x-2 = \frac{y-4}{-5} = \frac{z+3}{4}$$

Ex. 4 When does the above line intersect the xy-plane?

$z = \emptyset$

$$\frac{x-2}{1} = \frac{y-4}{-5} = \frac{3}{4}$$

$$x-2 = \frac{3}{4} \rightarrow x = \frac{11}{4} \qquad \text{intersection at } \left(\frac{11}{4}, \frac{1}{4}, \emptyset\right)$$

$$\frac{y-4}{-5} = \frac{3}{4} \rightarrow y = \frac{1}{4}$$

Distance between a Point and a Line

The distance between a line, defined by point P_0 & vector L, and a point Q is

$$\text{distance} = \frac{|\overrightarrow{PQ} \times \bar{L}|}{|\bar{L}|}$$

Ex. 5 Find the distance between the line defined by $P_0 = (3,2,1)$ & vector $\bar{L} = \langle -1,-2,-5\rangle$, and point $Q = (4,-1,8)$

$$\overline{P_0Q} = \langle 1,-3,7\rangle$$

$$\overline{P_0Q} \times \bar{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 7 \\ -1 & -2 & -5 \end{vmatrix} = \hat{i}[(-3\cdot -5) - (7\cdot -2)] + -\hat{j}[1\cdot -5 - 7\cdot -1] + \hat{k}[(1\cdot -2) - (-3\cdot -1)] = \hat{i}\,29 + -\hat{j}\,2 + \hat{k}\,-5 = \langle 29,-2,-5\rangle$$

$$|\overline{P_0Q} \times \bar{L}| = \sqrt{870}$$

$$|\bar{L}| = \sqrt{30} \qquad \text{dist} = \frac{\sqrt{870}}{\sqrt{30}} = \sqrt{29}$$

Ex. 6 Show that the lines L_1 & L_2 are skew lines.

$\bar{L}_1$: $x = 1+t \quad y = -2+3t \quad z = 4-t$

$\bar{L}_2$: $x = 2s \quad y = 3+s \quad z = -3+4s$

$\bar{V}_1 = \langle 1, 3, -1 \rangle$ & $\bar{V}_2 = \langle 2, 1, 4 \rangle \rightarrow$ Not $\parallel$

If they intersect $\exists t, s$ s.t. (there exists)

$$1+t = 2s \longrightarrow s = \frac{1+t}{2}$$

$$-2+3t = 3+s$$

$$4-t = -3+4s$$

$$-2+3t = 3 + \frac{1+t}{2}$$

$$-2+3t = \frac{7}{2} + \frac{t}{2}$$

$$-\frac{11}{2} + 3t = \frac{t}{2} \rightarrow -\frac{11}{2} = -\frac{5}{2}t \rightarrow t = \frac{11}{5} \text{ \& } s = \frac{8}{5}$$

$$4 - \frac{11}{5} \neq -3 + 4\left(\frac{8}{5}\right)$$

They do not intersect

Ex. 7 Find the intersection of the line defined by $\bar{P}_0 = \langle \frac{1}{2}, -\frac{1}{2}, \frac{1}{3} \rangle$ and $\bar{L} = \langle 1, 4, 2 \rangle$ and the sphere given by $x^2+y^2+z^2 = 1$

$$x = \frac{1}{2} + t \quad y = -\frac{1}{2} + 4t \quad z = \frac{1}{3} + 2t$$

$$\left(\frac{1}{2}+t\right)^2 + \left(-\frac{1}{2}+4t\right)^2 + \left(\frac{1}{3}+2t\right)^2 = 1$$

$$\frac{1}{4} + t + t^2 + \frac{1}{4} - 4t + 16t^2 + \frac{1}{9} + \frac{4}{3}t + 4t^2 = 1$$

$$21t^2 - \frac{5}{3}t + \frac{11}{18} - 1 = 0$$

$$21t^2 - \frac{5}{3}t - \frac{7}{18} = 0$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow t_1 = \frac{5}{126} - \frac{\sqrt{319}}{126} \qquad t_2 = \frac{5}{126} + \frac{\sqrt{319}}{126}$$