Lecture 31

Stoke's Theorem

We've already looked at Green's Theorem which related an integral over a plane region to a line integral around the boundary. This is just a special case of Stoke's Theorem.

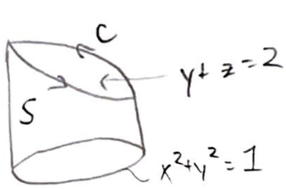
Let Σ be an oriented piecewise smooth surface bounded by a simple piecewise smooth boundary curve C that is positively oriented. Let F be a vector field whose components have continuous partial derivatives on Σ. Then,

$$\int_C \bar{F}\cdot d\bar{r} = \iint_\Sigma \operatorname{curl}(\bar{F})\cdot d\bar{S}$$

We often denote the positively oriented boundary curve of Σ as $\partial\Sigma$. So we write Stoke's Theorem as,

$$\int_{\partial\Sigma} \bar{F}\cdot d\bar{r} = \iint_\Sigma \operatorname{curl}(\bar{F})\cdot d\bar{S}$$

Ex. 1 Evaluate $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = \langle -y^2, x, z^2\rangle$ and C is the curve of the intersection of the plane $y+z=2$ and $x^2+y^2=1$.



$$\int_C \bar{F}\cdot d\bar{r} = \iint \operatorname{curl}(\bar{F})\cdot dS$$

$$\operatorname{curl}(\bar{F}) = \langle 0, 0, 1+2y\rangle$$

When the surface S is a graph we can write

$$\iint_S \bar{G}\cdot d\bar{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

when $\bar{G} = \langle P, Q, R\rangle$ and S is the graph $z = g(x,y)$

So,

$$\iint_S \text{curl}(\bar{F}) \cdot d\bar{S} = \iint_D (1+2y)\, dA$$

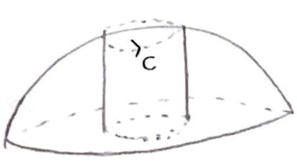
$$= \int_0^{2\pi} \int_0^1 (1 + 2r\sin(\theta))\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} + \frac{2}{3} r^3 \sin(\theta) \right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} + \frac{2}{3}\sin(\theta) \right) d\theta$$

$$= \frac{1}{2}(2\pi) + 0 = \pi$$

<u>Ex. 2</u> Evaluate $\iint_S \text{curl}(\bar{F}) \cdot d\bar{S}$ where $\bar{F} = \langle xz, yz, xy \rangle$ and S is the part of the sphere $x^2+y^2+z^2 = 4$ that lies in the cylinder $x^2+y^2=1$ above the xy-plane.



We will use Stoke's Theorem in the other direction

$$\iint_S \text{curl}(\bar{F}) \cdot d\bar{S} = \int_C \bar{F} \cdot d\bar{r}$$

The boundary C is described by the equations $x^2+y^2=1$ and $z = \sqrt{3}$.

$$\begin{array}{rl} x^2+y^2+z^2 &= 4 \\ -(x^2+y^2 &= 1) \\ \hline z^2 = 3 \rightarrow z = \sqrt{3} & \text{since } z > 0. \end{array}$$

Thus C is given by

$\bar{r}(t) = \langle \cos(t), \sin(t), \sqrt{3} \rangle$

$\bar{r}'(t) = \langle -\sin(t), \cos(t), 0 \rangle$

And,

$$\bar{F}(r(t)) = \langle \sqrt{3}\cos(t), \sqrt{3}\sin(t), \cos(t)\sin(t) \rangle$$

Thus,

$$\iint_S \text{curl}(\bar{F})\cdot d\bar{S} = \int_C \bar{F}\cdot d\bar{r} = \int_0^{2\pi} \bar{F}(\bar{r}(t))\cdot \bar{r}'(t)\, dt$$

$$= \int_0^{2\pi} \left(-\sqrt{3}\cos(t)\sin(t) + \sqrt{3}\sin(t)\cos(t)\right) dt$$

$$= 0.$$

<u>Surface Independence</u>

If a boundary, $\partial\Sigma$, is shared between two surfaces, Σ_1 & Σ_2, Stokes theorem tells us,

$$\iint_{\Sigma_1} \text{curl}(\bar{F})\cdot \bar{n}_1\, dS = \int_C \bar{F}\cdot d\bar{r} = \iint_{\Sigma_2} \text{curl}(\bar{F})\cdot \bar{n}_2\, dS$$