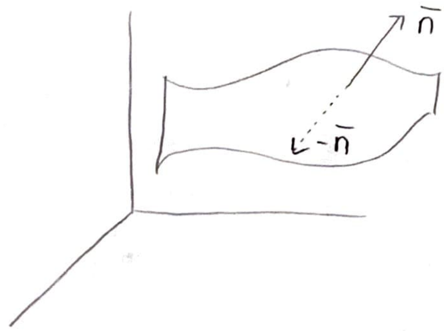
Lecture 30

Oriented Surfaces

For our next type of integration, we will need to formalize our understanding of surface orientation. Every point of a surface has two normal vectors pointing in opposite directions. If the surface is given by a parametric equation, $\bar{r}(u,v)$, the unit normal vector is given by:

$$\bar{n} = \frac{\bar{r}_u \times \bar{r}_v}{|\bar{r}_u \times \bar{r}_v|}.$$



If it is possible to pick a unit normal vector at every point, (x,y,z) such that $\bar{n}$ varies continuously over the surface, Σ, we call Σ an oriented surface and the choice of $\bar{n}$ gives Σ an orientation. These surfaces are sometimes called "two-sided". An example of a "one-sided" surface is the Möbius strip.

A closed surface is typically oriented such that the normal vector point "out". This is also called the "positive orientation".

Surface Integral of Vector Fields

Suppose Σ is an oriented surface with unit normal, $\bar{n}$. Imagine a fluid with density, $\rho(x,y,z)$, and velocity, $\bar{v}(x,y,z)$, is flowing through Σ. The rate of flow at a point is $\rho\bar{v}$. The mass of fluid flow in the direction of $\bar{n}$ is $(\rho\bar{v}\cdot\bar{n})$. If we divide Σ into many small regions, S_{ij}, the flow rate through any is given by $(\rho\bar{v}\cdot\bar{n})A(S_{ij})$. Taking the usual limit we get

$$\iint_{\Sigma} \rho\bar{v}\cdot\bar{n}\,dS = \iint \rho(x,y,z)\big(\bar{v}(x,y,z)\cdot\bar{n}(x,y,z)\big)d\bar{S}.$$

Physically, this is the rate of flow through Σ.

In general, an integral of the form,

$$\iint_{\Sigma} \bar{F} \cdot \bar{n}\, dS, \qquad \left(\iint_{\Sigma} \bar{F} \cdot d\bar{S}\right)$$

is called a "flux integral". Flux is latin for flow.

To evaluate these integrals, we typically use:

$$\iint_{\Sigma} \bar{F} \cdot \bar{n}\, dS = \pm \iint F(x(u,v), y(u,v), z(u,v)) \cdot [r_u(u,v) \times r_v(u,v)]\, dA.$$

Here, the $+$ is used if $\bar{r}_u \times \bar{r}_v$ is in the direction of $\bar{n}$. If $\bar{r}_u \times \bar{r}_v$ is in the opposite direction, we use the $-$.

Ex. 1 Find the flux of $\bar{F} = \langle z, y, x \rangle$ across the unit sphere, $x^2 + y^2 + z^2 = 1$.

$$r(\varphi, \theta) = \langle \sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi) \rangle$$

$$\bar{r}_\varphi \times \bar{r}_\theta = \langle \sin^2(\varphi)\cos(\theta), \sin^2(\varphi)\sin(\theta), \sin(\varphi)\cos(\varphi) \rangle$$

$$\bar{F}(r(\varphi,\theta)) \cdot (\bar{r}_\varphi \times \bar{r}_\theta) = \langle \cos(\varphi), \sin(\varphi)\sin(\theta), \sin(\varphi)\cos(\theta) \rangle \cdot \langle \sin^2(\varphi)\cos(\theta), \sin^2(\varphi)\sin(\theta), \sin(\varphi)\cos(\varphi) \rangle$$

$$= \cos(\varphi)\sin^2(\varphi)\cos(\theta) + \sin^3(\varphi)\sin^2(\theta) + \sin^2(\varphi)\cos(\varphi)\cos(\theta)$$

Thus,

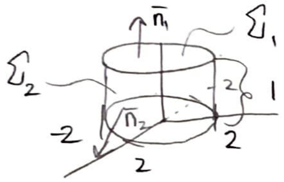
$$\iint \bar{F} \cdot d\bar{S} = \int_0^{2\pi}\int_0^{\pi} \left(2\sin^2(\varphi)\cos(\varphi)\cos(\theta) + \sin^3(\varphi)\sin^2(\theta)\right) d\varphi\, d\theta$$

$$= \frac{4\pi}{3}$$

If a surface, Σ, can be divided into several oriented surfaces, $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$, then we can define the integral as:

$$\iint_{\Sigma} \bar{F}\cdot\bar{n}\,dS = \iint_{\Sigma_1} \bar{F}\cdot\bar{n}\,dS + \dots + \iint_{\Sigma_n} \bar{F}\cdot\bar{n}\,dS$$

Ex. 2 Let Σ be the surface consisting of a circular top and cylindrical sides shown. Evaluate $\iint_{\Sigma} \bar{F}\cdot\bar{n}\,dS$ when $\bar{F} = \langle x, 0, z\rangle$.



$$\iint_{\Sigma} \bar{F}\cdot\bar{n}\,dS = \iint_{\Sigma_1} \bar{F}\cdot\bar{n}_1\,dS + \iint_{\Sigma_2} \bar{F}\cdot\bar{n}_2\,dS .$$

Σ_1: $\bar{n}_1 = \langle 0, 0, 1\rangle$

$\bar{F}\cdot\bar{n} = \langle x, 0, z\rangle\cdot\langle 0,0,1\rangle = z = 1$ on Σ_1

$$\iint_{\Sigma_1} 1\,dS = 4\pi$$

Σ_2: $\bar{r}(\theta, z) = \langle 2\cos(\theta), 2\sin(\theta), z\rangle$ $\quad 0 \le \theta \le 2\pi$ & $0 \le z \le 1$

$$\bar{n}_2 = \bar{r}_\theta \times \bar{r}_z = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & k \\ -2\sin(\theta) & 2\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2\cos(\theta), 2\sin(\theta), 0\rangle$$

with $0 \le \theta \le 2\pi$, this always points outwards.

$$\iint_{\Sigma_2} \bar{F}\cdot\bar{n}_2\,dS = \int_0^{2\pi}\int_0^1 \langle 2\cos(\theta), 0, z\rangle\cdot\langle 2\cos(\theta), 2\sin(\theta), 0\rangle\,dz\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 4\cos^2(\theta)\,dz\,d\theta$$

$$= \int_0^{2\pi} (2 + 2\cos(2\theta))\,d\theta = \left[2\theta + \sin(2\theta)\right]\Big|_0^{2\pi} = 4\pi$$

So,

$$\iint_{\Sigma} \bar{F}\cdot\bar{n}\,dS = 4\pi + 4\pi = 8\pi$$