Vector Multiplication

There are two ways mathematicians define vector multiplication. The first is the "dot product"

The Dot Product

The dot, inner, or scalar product is reatively simple. For two vector, $\bar{u}$ and $\bar{v}$ the inner product is

$$\bar{u}\cdot\bar{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Ex. 1 Find the inner product of $\bar{u} = \langle 1, 5, -3 \rangle$ & $\bar{v} = \langle 2, 8, 1 \rangle$

$$\bar{u}\cdot\bar{v} = \langle 1,5,-3\rangle \cdot \langle 2,8,1\rangle$$
$$= (1\cdot 2)+(5\cdot 8)+(-3\cdot 1) = 39$$

Useful Properties

$$\bar{u}\cdot\bar{v} = \bar{v}\cdot\bar{u}$$
$$\bar{u}\cdot(\bar{v}+\bar{w}) = \bar{u}\cdot\bar{v} + \bar{u}\cdot\bar{w}$$
$$(c\bar{u})\cdot\bar{v} = c(\bar{u}\cdot\bar{v}) = \bar{u}\cdot(c\bar{w})$$
$$(\bar{u}+\bar{v})\cdot\bar{w} = \bar{u}\cdot\bar{v} + \bar{v}\cdot\bar{w}$$

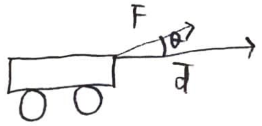
The dot product has an alternative definition.

$$\bar{u}\cdot\bar{v} = |\bar{u}||\bar{v}|\cos(\theta)$$

where θ is betwee 0° & 180° and is the angle between the two vectors.

ex. 2 A wagon is pulled 100m by a 70N force applied at 35° above horizontal. What is the work done?

$$W = \bar{F}\cdot\bar{d} = |\bar{F}||\bar{d}|\cos(\theta) = (70N)(100m)\cos(35^\circ) \approx 5734 Nm$$
$$= 5734 J$$



Test for Orthogonality

Two vectors are perpendicular iff their dot product is 0.

$$\bar{u} \perp \bar{v} \quad \text{iff} \quad \bar{u}\cdot\bar{v} = 0$$

Ex. 3 Show that $2\hat{i}+2\hat{j}-\hat{k}$ and $5\hat{i}-4\hat{j}+2\hat{k}$ are orthogonal.

$$(2\hat{i}+2\hat{j}-\hat{k})\cdot(5\hat{i}-4\hat{j}+2\hat{k}) = 2\cdot 5+2(-4)+(-1)2$$

$$= 10-8-2 = \emptyset$$

$$\therefore \perp$$

Ex. 4 Find the angle between $\bar{u} = \langle 2,2,-1\rangle$ & $\bar{v} = \langle 5,-3,2\rangle$

$$\bar{u}\cdot\bar{v} = |\bar{u}||\bar{v}|\cos(\theta)$$

$$\cos(\theta) = \frac{\bar{u}\cdot\bar{v}}{|\bar{u}||\bar{v}|}$$

$$\bar{u}\cdot\bar{v} = \langle 2,2,-1\rangle\cdot\langle 5,-3,2\rangle = 10-6-2 = 2$$

$$|\bar{u}| = \sqrt{2^2+2^2+(-1)^2} = 3$$

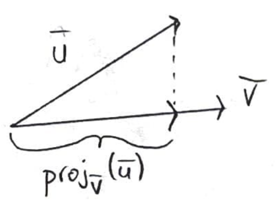
$$|\bar{v}| = \sqrt{5^2+(-3)^2+(2)^2} = \sqrt{38}$$

$$\cos(\theta) = \frac{2}{3\cdot\sqrt{38}}$$

$$\theta = \cos^{-1}\left(\frac{2}{3\cdot\sqrt{38}}\right) \approx 1.46 \approx 84^\circ$$

Projections

It is often useful to look at projections of one vector onto another.



$$\text{proj}_{\bar{v}}(\bar{u}) = \left(\frac{\bar{v}\cdot\bar{u}}{|\bar{v}|}\right)\frac{\bar{v}}{|\bar{v}|} = \frac{\bar{v}\cdot\bar{u}}{|\bar{v}|^2}\,\bar{v}$$ vector projection

The length of this vector projection is the scalar projection

$$\text{comp}_{\bar{v}}(\bar{u}) = \frac{\bar{v}\cdot\bar{u}}{|\bar{v}|}$$

Ex. 5 Find the scalar and vector projection of $\bar{u} = \langle 1,1,2 \rangle$ on to $\bar{v} = \langle -2,3,1 \rangle$

$$\text{comp}_{\bar{v}}(\bar{u}) = \frac{\bar{u}\cdot\bar{v}}{|\bar{v}|} = \frac{(1\cdot -2)+(1\cdot 3)+(2\cdot 1)}{\sqrt{(-2)^2+3^2+1^2}} = \frac{-2+3+2}{\sqrt{14}} = \frac{3}{\sqrt{14}}$$

$$\text{proj}_{\bar{v}}(\bar{u}) = \frac{\bar{u}\cdot\bar{v}}{|\bar{v}|}\frac{\bar{v}}{|\bar{v}|} = \frac{3}{\sqrt{14}}\frac{\langle -2,3,1\rangle}{\sqrt{14}} = \frac{3\langle -2,3,1\rangle}{14}$$

$$\text{proj}_{\bar{v}}(\bar{u}) = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$

The Cross Product

The second kind of vector multiplication is the "cross product". This is more complicated to compute.

$$\begin{aligned} \bar{u}\times\bar{v} = &(u_2v_3 - u_3v_2)\hat{i} + \\ &-(u_1v_3 - u_3v_1)\hat{j} + \\ &+(u_1v_2 - u_2v_1)\hat{k} \end{aligned}$$

Note the minus

I prefer using the "determinant" method.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\hat{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\hat{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\hat{k}$$

$$\begin{aligned} = &(u_2v_3 - u_3v_2)\hat{i} + \\ &-(u_1v_3 - u_3v_1)\hat{j} + \\ &+(u_1v_2 - u_2v_1)\hat{k} \end{aligned}$$

Ex. 6 Given $\bar{u} = \langle 1,3,4 \rangle$ & $\bar{v} = \langle 2,7,-5 \rangle$ find $\bar{u}\times\bar{v}$ & $\bar{v}\times\bar{u}$

$$\bar{u}\times\bar{v} \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = [(3\cdot -5)-(4\cdot 7)]\hat{i} - [(1\cdot -5)-4\cdot 2]\hat{j} + [(1\cdot 7)-(3\cdot 2)]\hat{k}$$
$$= (-15-28)\hat{i} - (-5-8)\hat{j} + (7-6)\hat{k} = \boxed{-43\hat{i} + 13\hat{j} + \hat{k}}$$

$$\bar{v}\times\bar{u} \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 7 & -5 \\ 1 & 3 & 4 \end{vmatrix} = [(7\cdot 4)-(-5\cdot 3)]\hat{i} - [(2\cdot 4)-(-5\cdot 1)]\hat{j} + [(2\cdot 3)-(7\cdot 1)]\hat{k}$$
$$= (28+15)\hat{i} - (8+5)\hat{j} + (6-7)\hat{k} = \boxed{43\hat{i} - 13\hat{j} - \hat{k}}$$

Useful Properties

$\bar{u} \times \bar{v} = -\bar{v} \times \bar{u}$ (anti-commutative)

$(c\bar{u}) \times \bar{v} = c(\bar{u} \times \bar{v}) = \bar{u} \times (c\bar{v})$

$\bar{u} \times (\bar{v} + \bar{w}) = \bar{u} \times \bar{v} + \bar{u} \times \bar{w}$

$(\bar{u} + \bar{v}) \times \bar{w} = \bar{u} \times \bar{w} + \bar{v} \times \bar{w}$

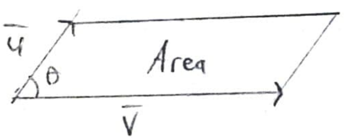
$\bar{u} \times \bar{u} = \emptyset$

Orthogonality and angles and areas

$\bar{u} \times \bar{v}$ is orthogonal to both $\bar{u}$ & $\bar{v}$. This can be used to create orthogonal vectors.

Also,

$|\bar{u} \times \bar{v}| = |\bar{u}||\bar{v}| \sin(\theta)$ = Area of parallelogram defined by $\bar{u}$ & $\bar{v}$



$\bar{u}$ & $\bar{v}$ are parallel iff $\bar{u} \times \bar{v} = \emptyset$

Scalar Triple Product

There are different ways to combine inner and cross products. One useful one is the scalar triple product:

$$\bar{u} \cdot (\bar{v} \times \bar{w})$$

This is important because this product finds the volume of a parallelopiped defined by $\bar{u}$, $\bar{v}$, & $\bar{w}$.

Ex. 7 Find the volume of the parallelopiped defined by $\bar{u} = \langle 5, 1, 3 \rangle$, $\bar{v} = \langle 2, 4, 3 \rangle$, & $\bar{w} = \langle 1, 3, 9 \rangle$

$$\bar{v} \times \bar{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 4 & 3 \\ 1 & 3 & 9 \end{vmatrix} = [(4 \cdot 9) - (3 \cdot 3)]\hat{\imath} - [(2 \cdot 9) - (3 \cdot 1)]\hat{\jmath} + [2 \cdot 3 - 4 \cdot 1]\hat{k}$$

$$= 27\hat{\imath} - 15\hat{\jmath} + 2\hat{k}$$

$$\bar{u} \cdot (\bar{v} \times \bar{w}) = \langle 5, 1, 3 \rangle \cdot \langle 27, -15, 2 \rangle = 126$$

$$V = |\bar{u} \cdot (\bar{v} \times \bar{w})| = 126$$