Lecture 29

We have already looked at integrals of piecewise smooth functions in the form of line integrals. We can extend this idea to integrals over piecewise smooth integrals.

Recall, we can parameterize a smooth surface, Σ, with a function of two parameters with a function such as:

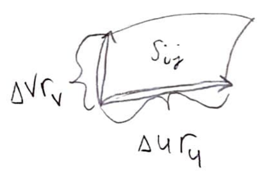
$$\bar{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle.$$

Surface Integrals

We have actually already seen a surface integral. The surface area was given by,

$$SA = \iint_S \|\bar{r}_u(u,v) \times \bar{r}_v(u,v)\| \, du \, dv.$$

Here, the area element is given by $\|\bar{r}_u \times \bar{r}_v\| du dv$. We can justify this with a picture



$$\text{Area of } S_{ij} \approx |(\Delta u \bar{r}_u) \times (\Delta v \bar{r}_v)|$$
$$\approx |\bar{r}_u \times \bar{r}_v| \Delta u \Delta v$$

when the region Σ is broken into an infinite number of subregions, S_{ij}, we get the above SA equation.

We might instead want to find the integral of a more general function over Σ. This extension is straightforward. The surface integral of a function $f(x,y,z)$ over a surface Σ parameterized by $\bar{r}(u,v)$, is defined by,

$$\iint_\Sigma f(x,y,z) \, dS = \iint_R f(x(u,v), y(u,v), z(u,v)) \, |r_u(u,v) \times r_v(u,v)| \, dA.$$

Note, if $f(x,y,z) = 1$ this gives the formula for the SA of Σ.

Ex. 1 Evaluate $\iint_S x^2 \, dS$ where S is the unit sphere.

Use the parameterization,

$$\bar{r}(\varphi,\theta) = \langle \sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi) \rangle$$

$$\bar{r}_\varphi = \langle \cos(\varphi)\cos(\theta), \cos(\varphi)\sin(\theta), -\sin(\varphi) \rangle$$

$$\bar{r}_\theta = \langle -\sin(\varphi)\sin(\theta), \sin(\varphi)\cos(\theta), \emptyset \rangle$$

$$\bar{r}_\varphi \times \bar{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos(\varphi)\cos(\theta) & \cos(\varphi)\sin(\theta) & -\sin(\varphi) \\ -\sin(\varphi)\sin(\theta) & \sin(\varphi)\cos(\theta) & \emptyset \end{vmatrix}$$

$$\begin{aligned} &= \hat{i}\left(\sin^2(\varphi)\cos(\theta)\right) + \\ &\quad -\hat{j}\left(-\sin^2(\varphi)\sin(\theta)\right) + \\ &\quad + \hat{k}\left(\cos^2(\theta)\cos(\varphi)\sin(\varphi) + \sin^2(\theta)\cos(\varphi)\sin(\varphi)\right) \end{aligned}$$

$$= \langle \sin^2(\varphi)\cos(\theta), \sin^2(\varphi)\sin(\theta), \cos(\varphi)\sin(\varphi) \rangle$$

$$|\bar{r}_\varphi \times r_\theta| = \sin(\varphi)$$

$$\iint x^2 \, dS = \iint \left(\sin(\varphi)\cos(\theta)\right)^2 |\bar{r}_\varphi \times \bar{r}_\theta| \, dA$$

$$= \int_\emptyset^{2\pi} \int_\emptyset^{\pi} \sin^2(\varphi)\cos^2(\theta)\sin(\varphi) \, d\varphi \, d\theta$$

$$= \int_\emptyset^{2\pi} \cos^2(\theta) \, d\theta \int_\emptyset^{\pi} \sin^3(\varphi) \, d\varphi$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]\Bigg|_\emptyset^{2\pi} \left[-\cos(\varphi) + \frac{1}{3}\cos^3(\varphi)\right]\Bigg|_\emptyset^{\pi}$$

$$= \frac{4\pi}{3}$$

If Σ is the graph of a function, $f(x,y)$, we can re-write the surface integral formula as,

$$\iint_{\Sigma} g(x,y,z)\,dS = \iint_{R} g(x,y,f(x,y)) \left[\sqrt{f_x^2 + f_y^2 + 1}\right] dA.$$

<u>Ex. 2</u> Evaluate $\iint_{\Sigma} y\,dS$ where Σ is the surface $z = x + y^2$ for $0 \le x \le 1$ & $0 \le y \le 2$.

$$f(x,y) = x + y^2$$

$$f_x = 1$$

$$f_y = 2y$$

$$\iint_{\Sigma} y\,dS = \iint_{R} y\sqrt{1^2 + (2y)^2 + 1}\,dA$$

$$= \int_0^2 \int_0^1 y\sqrt{2 + 4y^2}\,dx\,dy$$

$$= \int_0^2 y\sqrt{2 + 4y^2}\,dx\,dy$$

$$= \frac{1}{12}\left(2 + 4y^2\right)^{3/2}\Big|_0^2$$

$$= \frac{13\sqrt{2}}{3}$$