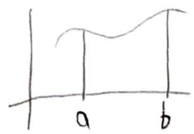
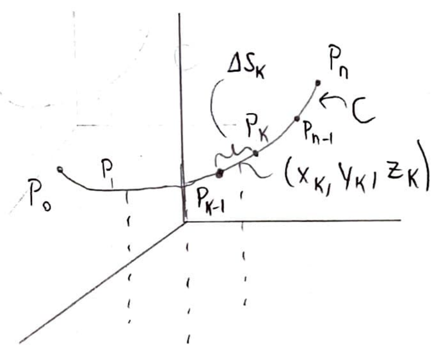
Lecture 26

Line Integrals

We have previously looked at integrals over intervals. For example, $\int_a^b f(x)\,dx$.



We can generalize this to integrating over arbitrary curves instead of intervals. The motivating example is how to find the mass of a finite wire with a given density $\rho(x,y,z)$.



Given a curve, C, we can "partition" it with partition P. This is similar to dividing an interval into many subrectangles. This leads us to,

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{K=1}^{n} f(x_K, y_K, z_K)\,\Delta S_K$$

Shorthand for $\|P\| \to \emptyset$

If C is a closed curve, we often use the notation

$$\oint_C f(x,y,z)\,ds$$

Computing a line integral in this way is difficult. We instead parameterize the curve.

If C is parameterized by a smooth VVF $\bar{r} = \langle x(t), y(t), z(t)\rangle$ with $t \in [a,b]$, we can compute a line integral using

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t)) \left|\frac{d\bar{r}}{dt}\right| dt$$

Ex. 1 Evaluate $\int_C 2x\,ds$ where C is the arc of $y=x^2$ from $(0,0)$ to $(1,1)$.

$$\bar{r} = \langle t, t^2 \rangle \quad t \in [0,1]$$

$$\int_C 2x\,ds = \int_0^1 2x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^1 2x\sqrt{1+4x^2}\,dt$$

$$= \frac{1}{4}\,\frac{2}{3}\left(1+4x^2\right)^{3/2}\Big|_0^1 = \frac{5\sqrt{5}-1}{6}$$

If C is piecewise smooth and composed of components $C_1, C_2, \ldots C_n$

$$\int_C f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \cdots \int_{C_n} f(x,y,z)\,ds$$

Line Integrals in Vector Fields

The work done on an object is given by $W = \bar{F} \cdot \overline{PQ}$. This formula works for objects moving in straight lines. If we want to know the work done on non-linear paths, we use line integrals.

In this case,

$$W = \int_C \bar{F}(x,y,z) \cdot \bar{T}(x,y,z)\,ds.$$

In general we write,

$$\int_C \bar{F} \cdot d\bar{r} = \int_C \bar{F}(x,y,z) \cdot \bar{T}(x,y,z)\,ds,$$

which is the line integral of F over C, where $\bar{T}$ is the tangent vector at (x,y,z).

To compute these we use,

$$\int_C \bar{F} \cdot d\bar{r} = \int_a^b \bar{F}(x(t), y(t), z(t)) \cdot \frac{d\bar{r}}{dt}\,dt$$

If $\bar{F} = \langle M, N, P \rangle$ we often write,

$$\int_C \bar{F}\cdot d\bar{r} = \int_C M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz$$

$$= \int_a^b \Big[M(x(t),y(t),z(t))\frac{dx}{dt} + $$
$$+ N(x(t),y(t),z(t))\frac{dy}{dt} +$$
$$+ P(x(t),y(t),z(t))\frac{dz}{dt}\Big]dt.$$

Ex. 3 Evaluate $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = \langle xy, yz, zx \rangle$ and C is the twisted cubic given by: $x=t$, $y=t^2$, $z=t^3$, $t\in[0,1]$

$$\bar{r} = \langle t, t^2, t^3 \rangle$$

$$\bar{r}' = \langle 1, 2t, 3t^2 \rangle$$

$$\bar{F}(t) = \langle t^3, t^5, t^4 \rangle$$

$$\bar{F}(t)\cdot\bar{r}'(t) = t^3 + 2t^6 + 3t^6 = t^3 + 5t^6$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 \bar{F}(t)\cdot\bar{r}'(t)dt = \int_0^1 (t^3 + 5t^6)\,dt = \frac{27}{28}$$

Orientation of C

$$\int_{-C} \bar{F}\cdot d\bar{r} = -\int_C \bar{F}\cdot d\bar{r}$$