Lecture 25

Vector Fields

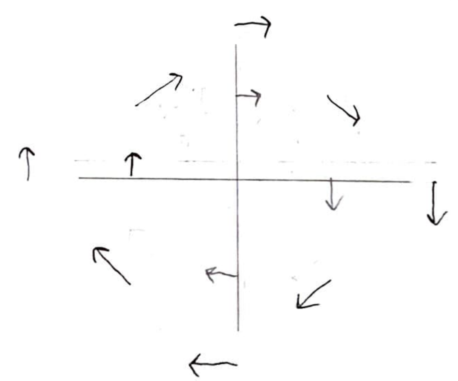
A vector field is a VVF that takes a point from its domain, D, and assigns it a vector.

$$\underbrace{\overline{F}(x,y,z)}_{\substack{\text{vector field} \\ \text{on } \mathbb{R}^3}} = \langle \underbrace{M(x,y,z),\ N(x,y,z),\ P(x,y,z)}_{\text{Component functions}} \rangle$$

Sometimes we write $\underbrace{\overline{x}}_{\text{position vector}} = \langle x, y, z \rangle$. Thus,

$$\overline{F}(x,y,z) = \overline{F}(\overline{x})$$

Ex. 1 Plot $\overline{F}(\overline{x}) = \langle \frac{y}{5}, -\frac{x}{5} \rangle$

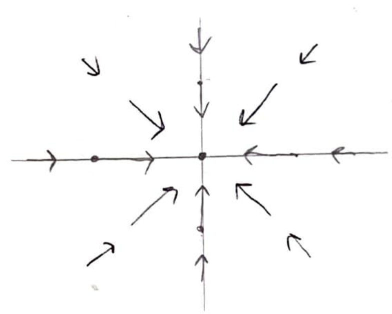


$\overline{F}(0,0) = \overline{0}$

$F(1,1) = \langle \frac{1}{5}, -\frac{1}{5} \rangle$

$F(1,0) = \langle 0, -\frac{1}{5} \rangle$

Ex. 2 Plot $\overline{F}(x,y,z) = \frac{-Gm}{(x^2+y^2+z^2)^{3/2}} \langle x, y, z \rangle$



We often write,

$$\overline{F}(\overline{r}) = \frac{-Gm}{|\overline{r}|^3} \overline{r}$$

Divergence

The divergence is given by

$$\operatorname{div}(\overline{F}(x,y,z)) = \overline{\nabla} \cdot \overline{F}(x,y,z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

It tells us the tendency of the vector field to diverge at a point.

Ex. 3 If $\bar{F}(x,y,z) = \langle xz, xyz, -y^2 \rangle$ compute $\operatorname{div}(\bar{F})$

$$\bar{\nabla}\cdot\bar{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}xyz + \frac{\partial}{\partial z}(-y^2)$$

$$\bar{\nabla}\cdot\bar{F} = z + xz$$

Curl

The curl is given by

$$\operatorname{curl}(\bar{F}(x,y,z)) = \bar{\nabla}\times\bar{F}$$

It tells us the tendency of a vector field to rotate.

Ex. 4 If $\bar{F}(x,y,z) = \langle xz, xyz, -y^2 \rangle$ compute $\operatorname{curl}(\bar{F})$

$$\bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} = \hat{i}(-2y - xy) + -\hat{j}(0 - x) + \hat{k}(yz - 0) = \langle -(2y+xy), x, yz \rangle$$

Conservative Vector Field

A vector field, $\bar{F}$, is conservative if $\exists$ some f s.t. $\bar{F} = \nabla f$. This f is called a potential function.

Ex. 5 If $f(x,y,z) = xy^2z$, find the conservative for which f is the potential function.

$$\bar{F} = \nabla f = \langle y^2 z, 2xyz, xy^2 \rangle$$

thus F is conservative

If $\bar{F}$ is conservative $\rightarrow$ $\bar{\nabla}\times\bar{F} = \bar{0}$

If $\bar{\nabla}\times\bar{F} \neq \bar{0} \rightarrow \bar{F}$ is not conservative.

If $\bar{\nabla}\times\bar{F} = \bar{0}$ + $\bar{F}$ is defined $\forall\ (x,y,z) \rightarrow \bar{F}$ is conservative.

Ex. 6

a) Show that $\bar{F}(x,y,z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$ is a cons. VF.

b) Find a potential function f s.t. $\bar{F} = \bar{\nabla} f$

a) $$\bar{\nabla} \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$= \langle 6xyz^2 - 6xyz^2, -(3y^2z^2 - 3y^2z^2), 2yz^3 - 2yz^3 \rangle$$

$$= \bar{\emptyset}$$

$\bar{\nabla} \times \bar{F} = \bar{\emptyset}$ & $\bar{F}$ is defined $\forall\ \mathbb{R}^3$

$\therefore$ $\bar{F}$ is conservative.

b) ① $f_x = y^2z^3$ $\xrightarrow{②}$ $f = xy^2z^3 + g(y,z)$

$f_y = 2xyz^3$ ↓ ③

$f_z = 3xy^2z^2$ ④ $f_y = 2xyz^3 + g_y(y,z) = 2xyz^3$

$\therefore\ g_y = \emptyset \rightarrow g(y,z) = h(z)$

$f_z = 3xy^2z^2 + h'(z) = 3xy^2z^2$

$\therefore\ h'(z) = \emptyset \rightarrow h(z) = K$

$f = xy^2z^3 + C$

We typically take $C = \emptyset$. So,

$f = xy^2z^3$