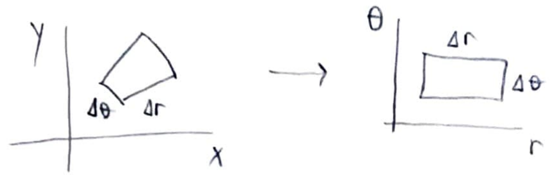
Lecture 23

Change of Variables



$x = r\cos(\theta)$

$y = r\sin(\theta)$

Area $= r\Delta r\Delta\theta$ Area $= \Delta r\Delta\theta$

This "warping" of space gave us:

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta$$

In general, if we change variables from (x,y) to (u,v) with the formulae

$$x = g_1(u,v) \quad \& \quad y = g_2(u,v)$$

where g_1 & g_2 are "continuously differentiable", the integral becomes:

$$\iint_R f(x,y)\,dA = \iint_S f(g_1(u,v), g_2(u,v))\,\underbrace{|J(u,v)|}_{\text{magnitude}}\,dA$$

where $J(u,v)$ is the Jacobian of the transformation, T, that converts R into S. The Jacobian is given by:

$$J(u,v) = \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}_{\text{determinant}}.$$

Ex. 1 Show the equation of an integral in polar coordinates is just a special case of the above formula.

$x = r\cos(\theta)$

$y = r\sin(\theta)$

$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}$$

$$J(u,v) = \cos(\theta)\, r\cos(\theta) - {}^{-}r\sin(\theta)\sin(\theta)$$
$$= r\cos^2(\theta) + r\sin^2(\theta)$$
$$J(u,v) = r$$

Thus,

$$\iint_R f(x,y)\,dA = \iint_S f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta.$$

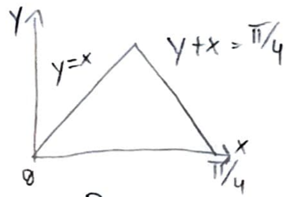
If we have formulae for u & v and <u>Not</u> x & y, we can use:

$$J(u,v) = \frac{1}{J(x,y)}, \quad \text{where } J(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}.$$

<u>Ex. 2</u> Evaluate $\displaystyle\iint_R \frac{\sin(x-y)}{\cos(x+y)}\,dA$ where R is the triangular region bounded by $y=\emptyset$, $y=x$, & $x+y=\pi/4$.

The form of the integral suggests,

$$u = x-y \quad \& \quad v = x+y$$



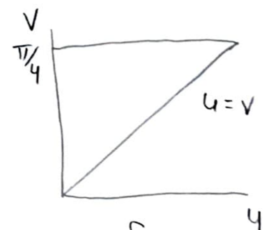
R

Now transform R,

$y=\emptyset$: $u = x = v \rightarrow v=u$

$y=x$: $u=\emptyset$

$x+y=\pi/4$: $v = x+y = \pi/4$



S

$$J(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1 - {}^{-}1 = 2$$

$$J(u,v) = \frac{1}{J(x,y)} = \frac{1}{2}$$

$$\iint_R \frac{\sin(x-y)}{\cos(x+y)}\,dA = \iint_S \frac{\sin(u)}{\cos(v)} J(u,v)\,dA = \int_0^{\pi/4}\int_0^{v} \frac{\sin(u)}{\cos(v)}\frac{1}{2}\,du\,dv$$

$$= \frac{1}{2}\int_0^{\pi/4} -\frac{\cos(u)}{\cos(v)}\Bigg|_0^{v} dv = \frac{-1}{2}\int_0^{\pi/4}\left(1-\frac{1}{\cos(v)}\right)dv$$

$$= \frac{-1}{2}\int_0^{\pi/4}(1-\sec(v))\,dv = \frac{1}{2}\Big[v - \ln(|\sec(v)+\tan(v)|)\Big]\Bigg|_0^{\pi/4}$$

$$= \frac{1}{2}\left(\ln(\sqrt{2}+1) - \frac{\pi}{4}\right)$$

In Three Dimensions

If we have,

$$x = g_1(u,v,w),\quad y = g_2(u,v,w),\quad z = g_3(u,v,w)$$

that define a transformation, T, that takes D to E we have:

$$\iiint_D f(x,y,z)\,dV = \iiint_E f(g_1(u,v,w), g_2(u,v,w), g_3(u,v,w))\, J(u,v,w)\,dA$$

where,

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Ex. 3 Derive the formula for integrals in spherical coordinates.

$$x = \rho\sin(\varphi)\cos(\theta) \qquad y = \rho\sin(\varphi)\sin(\theta) \qquad z = \rho\cos(\varphi)$$

$$J(\rho, \varphi, \theta) = \begin{vmatrix} \sin(\varphi)\cos(\theta) & \rho\cos(\varphi)\cos(\theta) & -\rho\sin(\varphi)\sin(\theta) \\ \sin(\varphi)\sin(\theta) & \rho\cos(\varphi)\sin(\theta) & \rho\sin(\varphi)\cos(\theta) \\ \cos(\theta) & -\rho\sin(\varphi) & \varnothing \end{vmatrix}$$

$$= \sin(\varphi)\cos(\theta)\left(\varnothing + \rho^2\sin^2(\varphi)\cos(\theta)\right) +$$

$$+ \rho\cos(\varphi)\cos(\theta)\left(\rho\sin(\varphi)\cos(\varphi)\cos(\theta)\right) +$$

$$- \rho\sin(\varphi)\sin(\theta)\left(-\rho\sin^2(\varphi)\sin(\theta) - \rho\cos^2(\varphi)\sin(\theta)\right)$$

$$= \rho^2\sin(\varphi)\left[\sin^2(\varphi)\cos^2(\theta) + \cos^2(\varphi)\cos^2(\theta) + \sin^2(\varphi)\sin^2(\theta) + \cos^2(\varphi)\sin^2(\theta)\right]$$

$$= \rho^2\sin(\varphi)$$