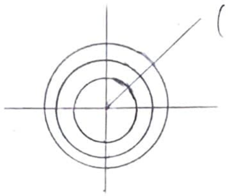
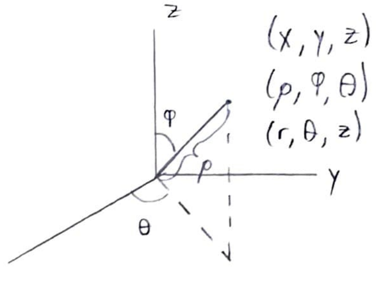
Lecture 22

Spherical Coordinates





$r = \rho \sin(\varphi)$ & $z = \rho \cos(\varphi)$

$x = r\cos(\theta)$ & $y = r\sin(\theta)$

$x = r\cos(\theta) = \rho \sin(\varphi)\cos(\theta)$

$y = r\sin(\theta) = \rho \sin(\varphi)\sin(\theta)$

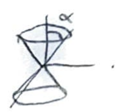
$z = \rho\cos(\varphi)$

$x^2 + y^2 = \rho^2 \sin^2(\varphi)$

$x^2 + y^2 + z^2 = \rho^2$

Ex. 1 a) What is the equation of a sphere w/ radius $= 4$ in spherical coordinates?

$\rho = 4$

b) Equation of a cone, .

$\varphi = \alpha$

c) The plane $z = 2$

$z = \rho\cos(\varphi)$

$\rho = \dfrac{2}{\cos(\varphi)} = 2\sec(\varphi)$

Integrating in Spherical Coordinates

If a region, D, can be described by the following equations:

$\alpha \le \theta \le \beta$

$h_1(\theta) \le \varphi \le h_2(\theta)$

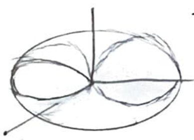
$F_1(\varphi,\theta) \le \rho \le F_2(\varphi,\theta)$

and if f is continuous, then,

$$\iiint_D f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{F_1(\varphi,\theta)}^{F_2(\varphi,\theta)} f(\rho\sin(\varphi)\cos(\theta), \rho\sin(\varphi)\sin(\theta), \rho\cos(\varphi))\,\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$

Note the factor of $\rho^2\sin(\varphi)$

Ex. 2 Find the volume of the torus $\rho = 3\sin(\varphi)$.



This is a horrible drawing of a "pinched torus"

$$V = \iiint_D dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{3\sin(\varphi)} \rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \frac{1}{3}\rho^3\sin(\varphi)\Big|_0^{3\sin(\varphi)} d\varphi$$

$$= 2\pi \int_0^{\pi} 9\sin^4(\varphi)\,d\varphi$$

$$= 18\pi \int_0^{\pi} \sin^4(\varphi)\,d\varphi = 18\pi\left[-\frac{1}{4}\sin^3(\varphi)\cos(\varphi) - \frac{3}{8}\sin(\varphi)\cos(\varphi) + \frac{3}{8}\varphi\right]\Bigg|_0^{\pi}$$

$$= 18\pi\left(\frac{3}{8}\pi\right) = \frac{27}{4}\pi^2$$

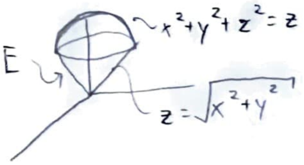
Ex. 3 Find the volume of the solid that lies above $z = \sqrt{x^2+y^2}$ and below $x^2+y^2+z^2 = z$.

$x^2+y^2+z^2 = z \rightarrow \rho^2 = \rho\cos(\varphi) \rightarrow \rho = \cos(\varphi)$ (a sphere)

$z = \sqrt{x^2+y^2} \rightarrow \rho\cos(\varphi) = \sqrt{\rho^2\sin^2(\varphi)\cos^2(\theta) + \rho^2\sin^2(\varphi)\sin^2(\theta)} = \rho\sin(\varphi)$

$\rightarrow \cos(\varphi) = \sin(\varphi)$

$\rightarrow \varphi = \pi/4$ (a cone)



$$V=\iiint_E dV=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos(\varphi)}\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$

$$=\int_0^{2\pi}d\theta\int_0^{\pi/4}\frac{1}{3}\rho^3\sin(\varphi)\Big|_0^{\cos(\varphi)}d\varphi$$

$$=2\pi\int_0^{\pi/4}\frac{1}{3}\sin(\varphi)\cos^3(\varphi)\,d\varphi \qquad u=\cos(\varphi)\quad du=-\sin(\varphi)\,d\varphi$$

$$=\frac{2}{3}\pi\left[-\frac{\cos^4(\theta)}{4}\right]\Bigg|_0^{\pi/4}=\frac{\pi}{8}$$