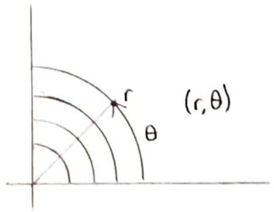
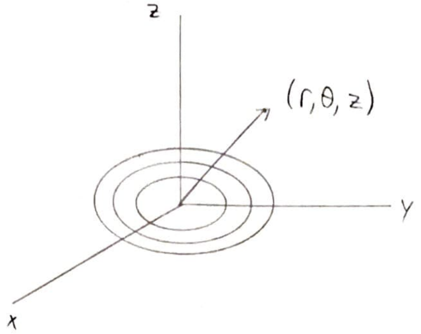
Lecture 21

Cylindrical Coordinates



$\longrightarrow$



$x^2+y^2=r^2 \qquad \tan(\theta)=\frac{y}{x} \qquad z=z$

$x=r\cos(\theta) \qquad y=r\sin(\theta) \qquad z=z$

	Rectangular	Cylindrical
Cylinder	$x^2+y^2=a^2$	$r=a$
Sphere	$x^2+y^2+z^2=a^2$	$r^2+z^2=a^2$
Double cone	$x^2+y^2=a^2z^2$	$r=az$ / $z=r\cot(\varphi_0)$
Circular paraboloid	$x^2+y^2=az$	$r^2=az$

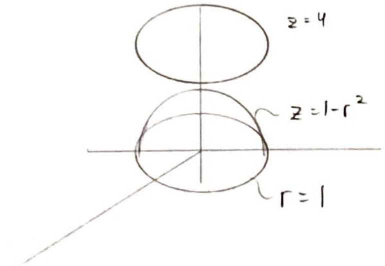
Cylindrical Integration

To compute an integral in cylindrical coordinates, we use,

$$\iiint_D f(x,y,z)\,dV=\int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)}\int_{F_1(r\cos(\theta),r\sin(\theta))}^{F_2(r\cos(\theta),r\sin(\theta))} f(r\cos(\theta),r\sin(\theta),z)\,r\,dz\,dr\,d\theta$$

Ex. 1

A solid, E, lies within the cylinder $x^2+y^2=1$, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.



$x^2+y^2=1 \rightarrow r=1$

$z=1-x^2-y^2 \rightarrow z=1-r^2$

$z = 1 - r^2$ to $z = 4$

$\theta = \emptyset$ to $\theta = 2\pi$

$r = \emptyset$ to $r = 1$

$$f(x,y,z) = K\sqrt{x^2+y^2} = Kr = \rho(x,y,z)$$

$$m = \iiint_E \rho(x,y,z)\,dV = \int_\emptyset^{2\pi}\int_\emptyset^{1}\int_{1-r^2}^{4} (Kr)r\,dz\,dr\,d\theta$$

$$m = \int_\emptyset^{2\pi} d\theta \int_\emptyset^{1} Kr^2\left[4-1+r^2\right]dr$$

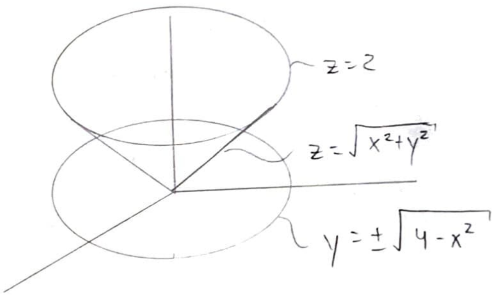
$$m = 2\pi K \int_\emptyset^{1} (3r^2 + r^4)\,dr$$

$$m = 2\pi K\left[r^3 + \frac{1}{5}r^5\right]\Bigg|_{r=\emptyset}^{r=1} = 2\pi K\left[1 + \tfrac{1}{5}\right] = \frac{12}{5}\pi K$$

Ex. 2

Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\,dz\,dy\,dx =$

$$= \int_\emptyset^{2\pi}\int_\emptyset^{2}\int_r^{2} r^2\, r\,dz\,dr\,d\theta$$



$$\int_0^{2\pi} d\theta \int_\emptyset^{2} r^3\left[2-r\right]dr = 2\pi\int_\emptyset^{2} 2r^3 - r^4\,dr$$

$$= 2\pi\left[\frac{1}{2}r^4 - \frac{1}{5}r^5\right]\Bigg|_{z=0}^{z=2} = 2\pi\,\frac{8}{5} = \frac{16}{5}\pi$$