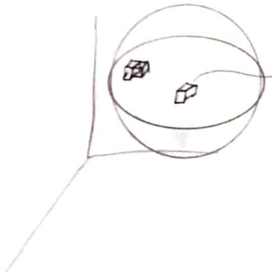
# Lecture 20

## Riemann Sum



parallelepiped

Volume: $\Delta V = \Delta x \Delta y \Delta z$

A function $f$ integrated over the region $D$ is computed as

$$\lim_{n\to\infty} \sum_{K=1}^{n} f(x_K, y_K, z_K)\,\Delta V_K = \iiint_D f(x,y,z)\,dV$$

## Triple Integral

If $D$ is the region between two continuous functions, $F_1$ & $F_2$ on a vertically or horizontally simple region, $R$, in the xy-plane and $f$ is continuous on $D$ then,

$$\iiint_D f(x,y,z)\,dV = \iint_R \left[\int_{F_1(x,y)}^{F_2(x,y)} f(x,y,z)\,dz\right] dA.$$

If,

a) $R$ is vertically simple,

$$\iiint_D f(x,y,z)\,dV = \int_a^b \left[\int_{B(x)}^{T(x)} \left[\int_{F_1(x,y)}^{F_2(x,y)} f(x,y,z)\,dz\right] dy\right] dx$$

b) $R$ is horizontally simple,

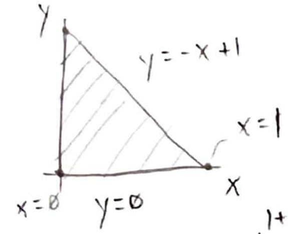
$$\iiint_D f(x,y,z)\,dV = \int_c^d \left[\int_{L(y)}^{R(y)} \left[\int_{F_1(x,y)}^{F_2(x,y)} f(x,y,z)\,dz\right] dx\right] dy$$

**Ex. 1**

Find the mass of $D$ which is the region between $z = x^2+y^2$ and $z = 1+x^2+y^2$ above the region, $R$, the triangle with corners $(0,0,0)$, $(0,1,0)$, and $(1,0,0)$, and the density of $D$ is $\rho(x,y,z) = xz$

The function we want to integrate is $\rho(x,y,z) = xz$.

R:




$$\int_{\emptyset}^{1}\int_{\emptyset}^{1-x}\int_{x^2+y^2}^{1+x^2+y^2} xz \, dz\, dy\, dx$$

$$x\int_{x^2+y^2}^{1+x^2+y^2} z\,dz = \frac{1}{2}x\, z^2\Big|_{x^2+y^2}^{1+x^2+y^2} = \frac{1}{2}x\left[(1+x^2+y^2)^2 - (x^2+y^2)^2\right] = \frac{1}{2}x\left[1+2x^2+2y^2\right] = \frac{1}{2}x + x^3 + xy^2$$

$$\int_0^{1-x} \frac{1}{2}x + x^3 + xy^2\, dy = \frac{1}{2}xy + x^3y + \frac{1}{3}xy^3\Big|_0^{1-x} = \frac{1}{2}x(1-x) + x^3(1-x) + \frac{1}{3}x(1-x)^3$$

$$= \frac{1}{6}x\left[-8x^3+12x^2-9x+5\right]$$

$$\int_0^1 \frac{1}{6}x\left[-8x^3+12x^2-9x+5\right]dx = \frac{1}{6}\int_0^1\left(-8x^4+12x^3-9x^2+5x\right)dx = \frac{1}{6}\left[-\frac{8}{5}x^5+3x^4-3x^3+\frac{5}{2}x^2\right]\Bigg|_{\emptyset}^{1}$$

$$\int_{\emptyset}^{1}\int_{\emptyset}^{1-x}\int_{x^2+y^2}^{1+x^2+y^2} xz\, dz\, dy\, dx = \frac{3}{2\emptyset}$$

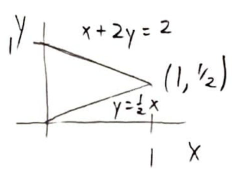
## Volume

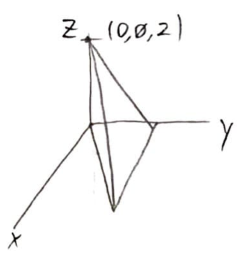
The Volume of a region D is given by

$$V = \iiint_D dV$$

## Ex. 2

Find the volume of the tetrahedron bounded by the planes:
$x+2y+z=2$, $x=2y$, $x=\emptyset$, and $z=\emptyset$




I hope this helps visualize it.

$$V = \int_{\emptyset}^{1} \int_{\frac{1}{2}x}^{1-\frac{1}{2}x} \int_{\emptyset}^{2-x-2y} dz\,dy\,dx$$

$$= \int_{\emptyset}^{1} \int_{\frac{1}{2}x}^{1-\frac{1}{2}x} 2-x-2y\,dy\,dx = \int_{\emptyset}^{1} 2y - xy - y^2 \Big|_{\frac{1}{2}x}^{1-\frac{1}{2}x} dx$$

$$= \int_{\emptyset}^{1} \left[(2-x)\left[1-\tfrac{1}{2}x - \tfrac{1}{2}x\right] - \left[(1-\tfrac{1}{2}x)^2 - \left(\tfrac{1}{2}x\right)^2\right]\right] dx$$

$$= \int_{\emptyset}^{1} (2-x)(1-x) - \left[1 + \tfrac{1}{4}x^2 - x - \tfrac{1}{4}x^2\right] dx$$

$$= \int_{\emptyset}^{1} (2-x)(1-x) - [1-x]\,dx$$

$$= \int_{\emptyset}^{1} 2 - 2x - x + x^2 - 1 + x\,dx$$

$$= \int_{\emptyset}^{1} 1 - 2x + x^2\,dx = x - x^2 + \tfrac{1}{3}x^3 \Big|_{\emptyset}^{1} = 1 - 1 + \tfrac{1}{3} = \tfrac{1}{3}$$