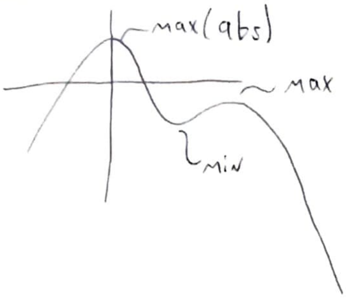
Lecture 15

Maxima & Minima in 2-D



In 2-D, a relative extreme value occurs at a critical point, $f'(x) = \emptyset$.

Extension to 3-D

The above condition can be extended for functions of two variables. If $f(x,y)$ has an extreme value at (x_0, y_0) then

$$f_x(x_0 y_0) = f_y(x_0, y_0) = \emptyset \quad \text{if} \quad f_x(x_0, y_0) \;\&\; f_y(x_0, y_0) \text{ exist}$$

$$\text{or} \quad \nabla f(x_0, y_0) = \bar{\emptyset}$$

These points (x_0, y_0) are called <u>critical points.</u>

Ex. 1

Let $f(x,y) = x^2 + y^2 - 2x - 6y + 14$. Find the critical points of f and determine any maxima or minima.

$$\nabla f = \langle f_x, f_y \rangle = \emptyset$$

$$f_x = 2x - 2 = \emptyset \qquad f_y = 2y - 6 = \emptyset$$

$$x = 1 \qquad y = 3$$

1 critical point at $(1, 3)$

$$f(x,y) = (x-1)^2 - 1 + (y-3)^2 - 9 + 14$$

$$f(x,y) = (x-1)^2 + (y-3)^2 + 4$$

$$(x-1)^2 \geq \emptyset \quad (y-3)^2 \geq \emptyset$$

$$f(x,y) \geq 4$$

$(1, 3)$ is a <u>global minimum</u>

Ex. 2

Let $f(x,y) = y^2 - x^2$. Find the critical points and any extrema.

$\nabla f(x,y) = \bar{0}$

$f_x = -2x = 0$ $\qquad$ $f_y = 2y = 0$

$x = 0$ $\qquad$ $y = 0$

$f(x,y) = y^2 - x^2$ is the hyperbolic paraboloid, please don't make me draw it. At the origin, moving in the y-direction $f(x,y)$ increases. In the x, it decreases. The origin is neither a max nor a min. It is a saddle point.

The Second Derivative Test

Assume f has a critical point at (x_0, y_0) and that f has continuous second derivatives in a disk centered at (x_0, y_0). Let

$$D(x_0,y_0) = f_{xx}(x_0,y_0)\, f_{yy}(x_0,y_0) - [f_{xy}(x_0,y_0)]^2$$

i) If $D(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) < 0$ (or $f_{yy}(x_0,y_0) < 0$), then f has a relative maximum at (x_0, y_0).

ii) If $D(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) > 0$ (or $f_{yy}(x_0,y_0) > 0$) then f has a relative minimum at (x_0, y_0)

iii) If $D(x_0,y_0) < 0$ then f has a saddle point at (x_0, y_0)

$D(x_0,y_0)$ is the discriminant of the Hessian matrix. That is,

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy} f_{yx} = f_{xx} f_{yy} - (f_{xy})^2$$

Ex. 3

Find the local maxima & minima of $f(x,y) = x^4 + y^4 - 4xy + 1$

$f_x = 4x^3 - 4y = 0$ $\qquad$ $f_y = 4y^3 - 4x = 0$

$x^3 - y = 0$ $\qquad$ $y^3 - x = 0$

$y = x^3$

$(x^3)^3 - x = 0$

$x^9 - x = 0$

$x(x^8 - 1) = 0 \rightarrow x(x^4+1)(x^4-1) = 0$

$\rightarrow x(x^4+1)(x^2+1)(x^2-1) = 0$

$x = 0, 1, -1$

critical points are

$(0,0)$, $(1,1)$, + $(-1,-1)$

Now we need the discriminate,

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

$f_x = 4x^3 - 4y$ $\qquad$ $f_y = 4y^3 - 4x$

$f_{xx} = 12x^2$ $\qquad$ $f_{yy} = 12y^2$

$f_{xy} = -4$

$D(x,y) = 144x^2y^2 - 16$

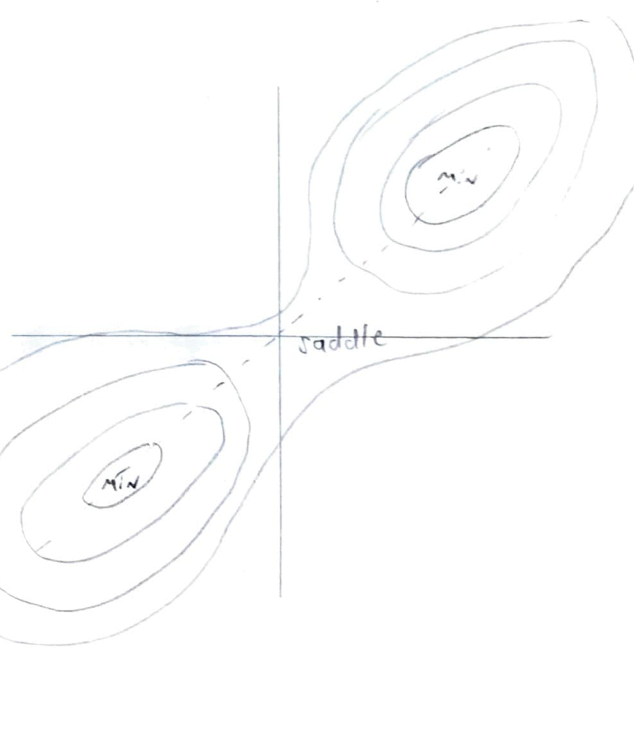
$(0,0)$: $D(0,0) = -16 < 0 \rightarrow$ saddle

$(1,1)$: $D(1,1) = 128 > 0$

$f_{xx}(1,1) = 12 > 0 \rightarrow$ local minimum

$(-1,-1)$: $D(-1,-1) = 128 > 0$

$f_{xx}(-1,-1) = 12 > 0 \rightarrow$ local minimum



Ex. 4

Let $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$. Find the extreme values.

$f_x = 6xy - 6x = 0$ $\qquad$ $f_y = 3x^2 + 3y^2 - 6y$

$6xy - 6x = 0$ $\qquad$ $3x^2 + 3y^2 - 6y = 0$

$xy - x = 0$ $\qquad$ $x^2 + y^2 - 2y = 0$

$x(y-1) = 0$ $\qquad$ $x^2 + y(y-2) = 0$

$x = 0$ $\qquad$ with $x = 0$

$y = 1$

$y(y-2) = 0$

$y = 0$ or $y = 2$

$\rightarrow \boxed{(0,0) \ \& \ (0,2)}$

with $y = 1$

$x^2 + 1(1-2) = 0$

$x^2 - 1 = 0$

$\rightarrow x = \pm 1$

$\boxed{(1,1) \ \& \ (-1,1)}$

Apply the second Derivative test

$f_{xx} = 6y - 6$ $\qquad$ $f_{yy} = 6y - 6$

$f_{xy} = 6x$

$D(x,y) = (6y-6)^2 - 36x^2$

$(0,0)$: $D(0,0) = 36 > 0$ & $f_{xx}(0,0) = -6 < 0 \rightarrow (0,0)$ is a max

$(0,2)$: $D(0,2) = 36 > 0$ & $f_{xx}(0,2) = 6 > 0 \rightarrow (0,2)$ is a min

$(-1,1)$: $D(-1,1) = -36 < 0 \rightarrow (-1,1)$ is a saddle point

$(1,1)$: $D(1,1) = -36 < 0 \rightarrow (1,1)$ is a saddle point