Lecture 14

Tangent Approximations in 2-D

In 2-D we write a secant approximation:

$$\frac{f(x)-f(x_0)}{x-x_0} \approx f'(x).$$

We can rearrange:

$$f(x) \approx f'(x)(x-x_0) + f(x_0).$$

Rearranging,

$$f(x) \approx y + f'(x)(x-x_0).$$

This is a first order approximation of $f(x)$. It is also the first two terms of the Taylor Series:

$$f(x) = y_0 + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \dots$$

Tangent Planes in 3-D

This extends pretty naturally to functions of two variables.

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Ex. 1 Approximate $\sqrt{(3.02)^2+6.95}$

Note this is approximately $\sqrt{3^2+7} = 4$.

$f(x,y) = \sqrt{x^2+y}$ $\quad x_0 = 3 \quad y_0 = 7$

$f_x = \frac{x}{\sqrt{x^2+y}}$ $\quad f_y = \frac{1}{2}\frac{1}{\sqrt{x^2+y}}$

$f_x(3,7) = \frac{3}{4}$ $\quad f_y(3,7) = \frac{1}{8}$

$$f(3.02, 6.95) = 4 + \frac{3}{4}(3.02-3) + \frac{1}{8}(6.95-7)$$

$$= 4 + \frac{3}{4}\left(\frac{1}{50}\right) - \frac{1}{8}\left(\frac{1}{20}\right)$$

$$\sqrt{(3.02)^2+6.95} = 4.00875$$

Extension to Functions of 3 Variables

The obvious extension is:

$$f(x,y,z) \approx f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0).$$

Ex. 2

The outer dimensions of a cardboard box are 14", 14", + 28". The cardboard is ¼" thick. What is the volume of the cardboard?

$$V(14,14,28) - V(14-\tfrac{1}{4}, 14-\tfrac{1}{4}, 28-\tfrac{1}{4}) \approx$$

$$\approx -V_x(14,14,28)(-\tfrac{1}{4}) - V_y(14,14,28)(-\tfrac{1}{4}) - V_z(14,14,28)(-\tfrac{1}{4})$$

$$\approx -(14\cdot 28)(-\tfrac{1}{4}) - (14\cdot 28)(-\tfrac{1}{4}) - (14\cdot 14)(-\tfrac{1}{4}) = 245$$

Differentials

We define the differential:

$$df = f_x(x,y)\,dx + f_y(x,y)\,dy$$

or

$$df = f_x(x,y,z)\,dx + f_y(x,y,z)\,dy + f_z(x,y,z)\,dz$$

Ex. 3

Two resistors, R_1 + R_2, are in parallel. $R_1 = 2\,\Omega$ + $R_2 = 6\,\Omega$. The tolerance in R_1 is $0.01\,\Omega$ + R_2 is 0.02. Estimate the max error in R.

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

$$dR = R_{R_1}\,dR_1 + R_{R_2}\,dR_2$$

$$R_1 = 0.01, \quad R_2 = 0.02, \quad R_{R_1} = \frac{R_2^2}{(R_1+R_2)^2}, \; + \; R_{R_2} = \frac{R_1^2}{(R_1+R_2)^2}$$

$$dR(2,6) = \frac{36}{64}(0.01) + \frac{4}{64}(0.02)$$

$$dR = 0.006875$$