Lecture 13

Recall, $f_x = \lim_{h\to 0} \dfrac{f(x+h,y)-f(x,y)}{h}$ & $f_y = \lim_{h\to 0} \dfrac{f(x,y+h)-f(x,y)}{h}$

These give the rate of change of f in the x & y directions. We sometimes want the r.o.c. in a different direction.

If we want the derivative in an arbitrary direction, $\hat{u} = \langle a,b\rangle$ we write:

$$D_u f(x,y) = \lim_{h\to 0} \frac{f(x+ha,\, y+hb) - f(x,y)}{h}.$$

To evaluate these we use:

$$D_u f(x,y) = f_x a + f_y b$$

In 3-D with $\hat{u} = \langle a, b, c\rangle$

$$D_u f(x,y) = f_x a + f_y b + f_z c$$

Ex. 1

With $f(x,y) = 6 - 3x^2 - y^2$ & $\bar{u} = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$, find $D_u f(1,2)$

Is $\bar{u}$ a unit vector?

$$|\bar{u}| = \sqrt{\left(\tfrac{1}{\sqrt{2}}\right)^2 + \left(\tfrac{-1}{\sqrt{2}}\right)^2} = 1$$

Yes it is!

$$f_x = -6x \qquad f_y = -2y$$

$$D_u f(x,y) = \langle f_x, f_y\rangle \cdot \hat{u} = -\frac{6}{\sqrt{2}}x + \frac{2}{\sqrt{2}}y$$

$$D_u f(1,2) = -\frac{6}{\sqrt{2}}(1) + \frac{2}{\sqrt{2}}(2) = -\sqrt{2}$$

Ex. 2

With $f(x,y) = x^3 - 3xy + 4y^2$ & $\hat{u}$ is the unit vector in the direction $\theta = \pi/6$, find $D_u f(1,2)$.

$$\hat{u} = \langle \cos(\pi/6),\ \sin(\pi/6)\rangle$$

$$f_x = 3x^2 - 3y \qquad f_y = -3x + 8y$$

$$D_u f(x,y) = (3x^2 - 3y)\cos(\pi/6) + (-3x + 8y)\sin(\pi/6)$$

$$D_u f(1,2) = \frac{13 - 3\sqrt{3}}{2}$$

The Gradient

Recall I wrote:

$$D_u f(x,y,z) = \langle f_x, f_y, f_z \rangle \cdot \langle a,b,c \rangle$$

This vector of first order partials is called the gradient. It is denoted as:

$$\nabla f = \langle f_x, f_y, f_z \rangle.$$

This is read as "del f", and ∇ is the del operator. The symbol is called Nabla.

We can rewrite,

$$D_u f(x,y) = \nabla f \cdot \hat{u}$$

which makes it clear $D_u f(x,y)$ is the projection of ∇f in the direction $\hat{u}$.

Ex. 3

With $f(x,y,z) = x\sin(yz)$ find ∇f and $D_v f(1,3,0)$ when $\bar{v} = \langle 1,2,-1 \rangle$

$$\nabla f = \langle \sin(yz), xz\cos(yz), xy\cos(yz) \rangle$$

$$\nabla f(1,3,0) = \langle 0, 0, 3 \rangle$$

Is $\bar{v}$ a unit vector?

$$|\bar{v}| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$$

$$\hat{v} = \frac{\bar{v}}{|\bar{v}|}$$

$$D_u f(1,3,0) = \langle 0,0,3 \rangle \cdot \langle \tfrac{1}{\sqrt{6}}, \tfrac{2}{\sqrt{6}}, -\tfrac{1}{\sqrt{6}} \rangle$$

$$D_u f(1,3,0) = -\sqrt{\tfrac{3}{2}}$$

Maximizing the Directional Derivative

The directional derivative is given by $|\nabla f(x)|$ and occurs in the direction of ∇f.

$D_u f = \nabla f \cdot \hat{u} = |\nabla f||\hat{u}|\cos(\theta) = |\nabla f|\cos(\theta)$

$\max(\cos(\theta)) = 1$ when $\theta = \emptyset$

So, $\max(D_u f) = |\nabla f|$ when $\nabla f \parallel \hat{u}$

Normal Properties

The gradient is normal to the function. If f is a function of two variables, $\nabla f(x_0, y_0)$ is normal to the level curve at (x_0, y_0). Similarly for 3 dimensions.

Ex. 4

Find a vector normal to $y = x^2$ @ $(3, 9)$.

$f(x,y) = y - x^2$

$\nabla f = \langle -2x, 1 \rangle$

$\nabla f(3,9) = \langle -6, 9 \rangle$

Ex. 5

Find an equation of the plane tangent to the sphere, $x^2 + y^2 + z^2 = 4$ @ $(-1, 1, \sqrt{2})$

$f(x,y,z) = x^2 + y^2 + z^2 = 4$

$\nabla f = \langle 2x, 2y, 2z \rangle$

$\nabla f(-1, 1, \sqrt{2}) = \langle -2, 2, 2\sqrt{2} \rangle$

$\bar{n} = \langle -2, 2, 2\sqrt{2} \rangle \quad P_0 = (-1, 1, \sqrt{2})$

$-2(x+1) + 2(y-1) + 2\sqrt{2}(z - \sqrt{2}) = \emptyset$